Math 210C Lecture 12 Notes

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1 Exact Sequences and the Snake Lemma

1.1 Exact sequences

Definition 1.1. A sequence in a category C is a collection $A_{\cdot} = ((A_i)_{i \in I}, (d_i^A)_{i \in I \cap (I+1)})$ of objects $A_i \in C$ and morphisms $d_i^A : A_i \to A_{i-1}$ for $i, i-1 \in I$, where I is the set of integers in an interval in \mathbb{R} .

$$\cdots \longrightarrow A_i \xrightarrow{d_i^A} A_{i-1} \xrightarrow{d_{i-1}^A} A_{i-2} \longrightarrow \cdots$$

Definition 1.2. A diagram

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in a category C with a zero object, equalizers, coequalizers, finite products, finite coproducts, and in which every morphism is strict, is **exact** if $g \circ f = 0$ and $\operatorname{im}(f) \to \operatorname{ker}(g)$ is an isomorphism.

Example 1.1. In Ab,

$$\mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z}$$

is not exact, but

 $\mathbb{Z} \xrightarrow{x \mapsto 2x} \mathbb{Z}/4\mathbb{Z} \xrightarrow{x \mapsto x} \mathbb{Z}/2\mathbb{Z}$

is exact.

Definition 1.3. A sequence A_{i} with defining interval I is **exact** if every subdiagram

$$A_{i+1} \xrightarrow{d_{i+1}} A_i \xrightarrow{d_i} A_{i-1}$$

is exact, where $i, i - 1, i + 1 \in I$.

Example 1.2. The sequence

$$\cdots \longrightarrow \mathbb{Q}^2 \xrightarrow{f} \mathbb{Q}^2 \xrightarrow{f} \mathbb{Q}^2 \xrightarrow{f} \mathbb{Q}^2 \xrightarrow{f} \cdots$$

is exact, where f(a,b) = (0,a) for al $(a,b) \in \mathbb{Q}^2$. This is a **long exact sequence**.

Definition 1.4. Let C be a category satisfying the aforementioned conditions. Let $A, B, C \in C$, and let $f : A \to B, g : B \to C$.

1. We say the diagram

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

is a left short exact sequence if it is exact.

2. We say the diagram

 $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$

is a **right short exact sequence** if it is exact.

3. We say the diagram

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a short exact sequence if it is exact.

Remark 1.1. In a short exact sequence, f is a monomorphism and g is an epimorphism.

Example 1.3. In Grp, if $N \leq G$, then

 $1 \longrightarrow N \longrightarrow G \longrightarrow G/N \longrightarrow 1$

is short exact.

Example 1.4. In Ab,

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

is short exact.

Definition 1.5. A long exact sequence is an exact sequence in which the interval of definition is of infinite length.

1.2 Morphisms of exact sequences and the snake lemma

Definition 1.6. Let A_i and B_i be sequence in C with defining intervals I and J. A **morphism** $f_i : A_i \to B_i$ is a sequence $(f)_{i \in O \cap J}$ of morphisms $f_i : A_i \to B_i$ such that $f_i \circ f_{i+1}^A = d_i^B \circ f_{i+1}$ for all $I \cap (I-1) \cap J \cap (J-1)$.

$$0 \xrightarrow{d_3^A} A_2 \xrightarrow{d_2^A} A_1 \xrightarrow{d_1^A} A_0$$
$$\downarrow^{f_2} \qquad \downarrow^{f_1} \qquad \downarrow^{f_0}$$
$$B_2 \xrightarrow{d_2^B} B_1 \xrightarrow{d_1^B} B_0 \xrightarrow{0} 0$$

Example 1.5. Here is a morphism between short exact sequences:

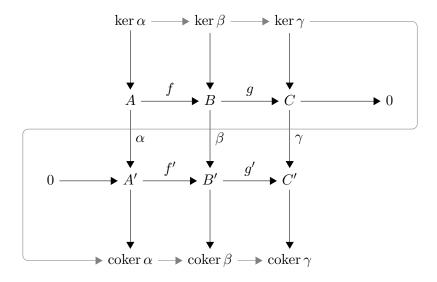
Lemma 1.1 (Snake lemma). Let C be an abelian category. Suppose we have a diagram

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$
$$\downarrow^{\alpha} \qquad \downarrow^{\beta} \qquad \downarrow^{\gamma}$$
$$0 \longrightarrow A' \xrightarrow{f'} B' \xrightarrow{g'} C'$$

in C which commutes and such that the two rows are exact. Then there is an exact sequence

$$\ker(\alpha) \to \ker(\beta) \to \ker(\gamma) \xrightarrow{\delta} \operatorname{coker}(\alpha) \to \operatorname{coker}(\beta) \to \operatorname{coker}(\gamma)$$

such that the following $diagram^1$ commutes:



¹The code for this diagram was modified from an answer on this StackExchange post.