Math 210C Lecture 12 Notes

Daniel Raban

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1 Exact Sequences and the Snake Lemma

1.1 Exact sequences

Definition 1.1. A **sequence** in a category $\mathcal{C}$ is a collection $A_\cdot = ((A_i)_{i \in I}, (d_i^A)_{i \in I \cap (I+1)})$ of objects $A_i \in \mathcal{C}$ and morphisms $d_i^A : A_i \to A_{i-1}$ for $i, i-1 \in I$, where I is the set of integers in an interval in $\mathbb{R}$.

$$\cdots \longrightarrow A_i \xrightarrow{d_i^A} A_{i-1} \xrightarrow{d_{i-1}^A} A_{i-2} \longrightarrow \cdots$$

Definition 1.2. A diagram

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in a category $\mathcal{C}$ with a zero object, equalizers, coequalizers, finite products, finite coproducts, and in which every morphism is strict, is **exact** if $g \circ f = 0$ and $\operatorname{im}(f) \to \ker(g)$ is an isomorphism.

Example 1.1. In Ab,

$$\mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z}$$

is not exact, but

$$\mathbb{Z} \xrightarrow{x \mapsto 2x} \mathbb{Z}/4\mathbb{Z} \xrightarrow{x \mapsto x} \mathbb{Z}/2\mathbb{Z}$$

is exact.

Definition 1.3. A sequence $A_\cdot$ with defining interval I is **exact** if every subdiagram

$$A_{i+1} \xrightarrow{d_{i+1}} A_i \xrightarrow{d_i} A_{i-1}$$

is exact, where $i, i-1, i+1 \in I$.

Example 1.2. The sequence

$$\cdots \longrightarrow \mathbb{Q}^2 \xrightarrow{f} \mathbb{Q}^2 \xrightarrow{f} \mathbb{Q}^2 \xrightarrow{f} \mathbb{Q}^2 \xrightarrow{f} \cdots$$

is exact, where $f(a, b) = (0, a)$ for al $(a, b) \in \mathbb{Q}^2$. This is a **long exact sequence**.

Definition 1.4. Let $\mathcal{C}$ be a category satisfying the aforementioned conditions. Let $A, B, C \in \mathcal{C}$, and let $f : A \to B$, $g : B \to C$.

1. We say the diagram
$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$
is a **left short exact sequence** if it is exact.

2. We say the diagram
$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$
is a **right short exact sequence** if it is exact.

3. We say the diagram
$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$
is a **short exact sequence** if it is exact.

Remark 1.1. In a short exact sequence, f is a monomorphism and g is an epimorphism.

Example 1.3. In Grp, if $N \trianglelefteq G$, then

$$1 \longrightarrow N \longrightarrow G \longrightarrow G/N \longrightarrow 1$$

is short exact.

Example 1.4. In Ab,

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

is short exact.

Definition 1.5. A **long exact sequence** is an exact sequence in which the interval of definition is of infinite length.

1.2 Morphisms of exact sequences and the snake lemma

Definition 1.6. Let $A_.$ and $B_.$ be sequence in $\mathcal{C}$ with defining intervals I and J. A **morphism** $f_. : A_. \to B_.$ is a sequence $(f)_{i \in O \cap J}$ of morphisms $f_i : A_i \to B_i$ such that $f_i \circ f_{i+1}^A = d_i^B \circ f_{i+1}$ for all $I \cap (I-1) \cap J \cap (J-1)$.

$$\begin{tikzcd}
0 \arrow[r, "d_3^A"] & A_2 \arrow[r, "d_2^A"] \arrow[d, "f_2"] & A_1 \arrow[r, "d_1^A"] \arrow[d, "f_1"] & A_0 \arrow[d, "f_0"] & \\
& B_2 \arrow[r, "d_2^B"] & B_1 \arrow[r, "d_1^B"] & B_0 \arrow[r, "0"] & 0
\end{tikzcd}$$

Example 1.5. Here is a morphism between short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "4"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}/4\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}/4\mathbb{Z} \arrow[r, "4"] & \mathbb{Z}/16\mathbb{Z} \arrow[r] & \mathbb{Z}/4\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Lemma 1.1 (Snake lemma)**.** *Let $\mathcal{C}$ be an abelian category. Suppose we have a diagram*

$$\begin{tikzcd}
& A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' &
\end{tikzcd}$$

in $\mathcal{C}$ which commutes and such that the two rows are exact. Then there is an exact sequence

$$\ker(\alpha) \to \ker(\beta) \to \ker(\gamma) \xrightarrow{\delta} \operatorname{coker}(\alpha) \to \operatorname{coker}(\beta) \to \operatorname{coker}(\gamma)$$

such that the following diagram[1] *commutes:*

$$
\begin{tikzcd}
 & \ker\alpha \arrow[r] \arrow[d] & \ker\beta \arrow[r] \arrow[d] & \ker\gamma \arrow[d] \arrow[dddll, rounded corners] & \\
 & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] \arrow[d] & B' \arrow[r, "g'"] \arrow[d] & C' \arrow[d] & \\
 & \operatorname{coker}\alpha \arrow[r] & \operatorname{coker}\beta \arrow[r] & \operatorname{coker}\gamma &
\end{tikzcd}
$$

[1] The code for this diagram was modified from an answer on this StackExchange post.